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# On the Nonuniqueness of Monosplines with Least $L_2$-Norm

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## 1

Let $\gamma(t, x)$ be an extended totally positive kernel [4] defined on $T \times X$, where $T$ and $X$ are intervals on the real line. Then a function of the form

$$F(x) = \sum_{\nu=1}^{r} \alpha_\nu \cdot \gamma(t_\nu, x), \qquad \alpha_\nu \in \mathbb{R}, \qquad t_\nu \in T, \qquad \nu = 1, 2, \ldots, r, \tag{1}$$

is called a $\gamma$-polynomial of order $r$ [2]. In connection with the determination of best quadrature formulas there arises the problem of approximating by $\gamma$-polynomials those functions in $L_p(X)$ that have a representation

$$f(x) = \int_T \gamma(t, x)\, d\mu(t), \tag{2}$$

$d\mu$ being a nonnegative measure. Then

$$m(x) = f(x) - F(x) \tag{3}$$

is called a monospline of order $r + 1$, if $F$ is a $\gamma$-polynomial of order $r$ [3, 5].

In this situation there is a unique solution of the nonlinear approximation problem, if the approximation is understood in the Chebyshev sense. But the question of uniqueness for the monospline of least $L_p$-norm was not settled for $1 < p < \infty$ [1, 5].

Our objective is to prove by an example that uniqueness does not hold in general.

## 2

Set $X = [-1, +1]$ and let $L_2(X)$ denote the space of square-integrable functions on $X$ endowed with the norm $\|f\| = (f, f)^{1/2}$, where

$$(f, g) = \int_{-1}^{+1} f(x)\, g(x)\, dx.$$

The kernel $\gamma(t, x) = (1 - tx)^{-1}$ is extended totally positive on

$$(-1, +1) \times [-1, +1].$$

Therefore,

$$f(x) = f_s(x) = \frac{1}{2}\left(\frac{1}{1 - sx} + \frac{1}{1 + sx}\right), \qquad 0 \leqslant s < 1, \tag{4}$$

admits a representation (2) with the measure $d\mu$ concentrated on the points $+s$ and $-s$. We consider the case $r = 1$. Then the approximation problem is equivalent to the minimization of the function

$$\rho(\alpha, t) = \left\|f - \frac{\alpha}{1 - tx}\right\|^2 = \left(f - \frac{\alpha}{1 - tx}, f - \frac{\alpha}{1 - tx}\right),$$
$$\alpha \in \mathbb{R}, \qquad -1 < t < +1. \tag{5}$$

Because of the symmetry relation $f(x) = f(-x)$ we know that optimality of $(\alpha, t)$ implies that $(\alpha, -t)$ is also optimal. Hence, if uniqueness is assumed, then the $t$-coordinate of the solution is zero. Moreover, if we fix $t$, then $\rho(\alpha, t)$ is quadratic in $\alpha$ and $\rho$ attains its minimum at

$$\alpha = \alpha_t = \frac{(f, (1 - tx)^{-1})}{((1 - tx)^{-1}, (1 - tx)^{-1})}. \tag{6}$$

Inserting (4) and $t = 0$, we obtain

$$\alpha_0 = \frac{\int_{-1}^{+1} 1 \cdot f \cdot dx}{\int_{-1}^{+1} 1 \cdot dx} = \frac{1}{2}\int_{-1}^{+1} \frac{1}{2}\left(\frac{1}{1 - sx} + \frac{1}{1 + sx}\right) dx$$
$$= \frac{1}{2s} \log \frac{1 + s}{1 - s} > 0. \tag{7}$$

The function $\rho$ is differentiable with respect to $t$. We have

$$\rho_t(\alpha, t) = -2\alpha \cdot (x(1 - tx)^{-2},\ f - \alpha(1 - tx)^{-1});$$
$$\rho_{tt}(\alpha, t) = 2\alpha \cdot (x^2 \cdot (1 - tx)^{-3},\ 3\alpha \cdot (1 - tx)^{-1} - 2f).$$

We check the second derivation at $(\alpha_0, 0)$:

$$\frac{1}{2\alpha_0} \cdot \rho_{tt}(\alpha_0, 0) = \int_{-1}^{+1} 3\alpha_0 x^2\, dx - \int_{-1}^{+1} x^2 \left(\frac{1}{1 - sx} + \frac{1}{1 + sx}\right) dx$$
$$= \left(\frac{1}{s} - \frac{2}{s^3}\right) \log \frac{1 + s}{1 - s} + \frac{4}{s^2},$$
$$< \frac{1}{s^2}\left(4 - \log \frac{1 + s}{1 - s}\right), \qquad 0 < s < 1. \tag{8}$$

The value is positive for small $s$, but it tends to $-\infty$, as $s$ tends to 1. In particular, it is negative for $s \geqslant 0.97$.

We conclude that for $s \geqslant 0.97$ the point $(\alpha_0, 0)$ is not a minimum, but only a saddle point of $\rho$. Hence, there are at least two monosplines of least $L_2$-norm. Moreover, we know that the best approximations are not symmetric functions.

The preceding investigation established nonuniqueness for a function $f$ that is represented according to (2) with $d\mu$ concentrated on two points. Observe that we have also nonuniqueness for all symmetric functions in a sufficiently small neighborhood of $f$. In particular, this neighborhood also contains functions of the form (2) for which the measure is not concentrated on a finite number of points.

## 3

A similar investigation can be performed for the extended totally positive kernel $\gamma(t, x) = e^{tx}$, $X = [-1, +1]$, $T = \mathbb{R}$ and the one-parameter family of functions

$$f(x) = f_s(x) = \tfrac{1}{2}(e^{sx} + e^{-sx}),\ s \geqslant 0.$$

With $\rho(\alpha, t) = \|f - \alpha \cdot e^{tx}\|^2$ we verify that $\rho_{tt}(\alpha_0, 0)$ is negative, whenever $s \geqslant 6$.

*Note added in proof.* A similar analysis may be performed for the Hilbert space of functions analytic in the unit disc and square integrable on its boundary. Put

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\varphi})|^2 \, d\varphi,$$

and consider the kernel from Section 2. For the approximation problem with $r = 1$ we have uniqueness if $f$ is a symmetric function and possesses a representation (2) with $d\mu$ being nonnegative and concentrated on the subinterval $[-\frac{1}{2}, +\frac{1}{2}]$. This result is sharp because we have nonuniqueness for $f_s$ whenever $\frac{1}{2} < s < 1$. — Extensions to unsymmetric functions and to $r \geqslant 2$ are not known.

## References


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